

# On the radicals of exponential Lie groups

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## Abstract

Let  $G$  be a connected exponential Lie group and  $R$  be the solvable radical of  $G$ . We describe a condition on  $G/R$  under which one can then conclude that  $R$  is an exponential Lie group. The condition holds in particular when  $G$  is a complex Lie group and this yields a stronger version of a result of Moskowitz and Sacksteder [9] on the center of a complex exponential Lie group being connected. Along the way we prove a criterion for elements from certain subsets of a solvable Lie group to be exponential, which would be of independent interest.

*Keywords:* Exponential Lie groups, solvable radical, connectedness of center.

## 1 Introduction

Let  $G$  be a connected Lie group,  $\mathfrak{G}$  be the Lie algebra of  $G$ , and let  $\exp : \mathfrak{G} \rightarrow G$  be the exponential map. An element  $g \in G$  is said to be *exponential* if there exists  $X \in \mathfrak{G}$  such that  $g = \exp X$ , namely if  $g$  is contained in the image of the exponential map. The Lie group  $G$  is said to be exponential if every element is exponential, or equivalently if the exponential map is surjective. There has been considerable interest in literature to understand which Lie groups are exponential. On the whole satisfactory results are known when  $G$  is either solvable or semisimple (see [5] for the status until the mid-1990's and [10], [11] and [12] for later work). A general Lie group is an almost semidirect product of a semisimple and a solvable subgroup (namely a semisimple Levi subgroup and the solvable radical respectively), but not much clarity is attained so far as to which semidirect products are exponential Lie groups. Some results in this respect may be found in [3], [4], [9] and [2].

In this context we consider the following question. Given an exponential Lie group  $G$  when can we conclude that its radical  $R$  is exponential? We describe a condition on  $G/R$ , the semisimple quotient Lie group, which enables such a

conclusion (see Theorem 1.2); the condition involved holds for a large class of semisimple Lie groups (see Theorem 1.3).

The question as above arose in the context of the following. The main theorem (Theorem 1) in [9] states that a connected complex Lie group  $G$  is exponential if and only if its center is connected and the adjoint group  $\text{Ad}(G)$  is exponential; as has been noted in the review of the paper in Mathematical Reviews the proof of the theorem given in the paper is valid only under an additional condition that  $G$  is “semi-algebraic” (we shall not go into the details of the condition as it does not concern us in the sequel). The “if” part, and also part of the converse, is relatively easy to see, so the main content of such a statement (when valid) is that the center of a connected complex exponential Lie group is connected; moreover, a complex semisimple exponential Lie group has trivial center, and hence conclusion is equivalent to that the center of the radical is connected. This would follow directly, if we conclude that the radical is exponential, since the center of an exponential solvable Lie group is always connected (see in particular, Corollary 2.9, *infra*). Our results apply to this case and in particular yield the above mentioned statement from [9] as a corollary in a special case. Our technique however is independent of [9].

We now formulate the condition involved and state the main results.

To begin, we recall that a one-parameter subgroup of a Lie group  $L$  is said to be *unipotent* if it is of the form  $\{\exp tX\}_{t \in \mathbb{R}}$ , where  $X$  is an element of the Lie algebra of  $L$  such that the adjoint transformation  $\text{ad}X$  is a nilpotent linear transformation of the Lie algebra.

**Definition 1.1.** We say that a connected Lie group  $L$  satisfies *condition  $\mathfrak{U}$*  if there exists a unipotent one-parameter subgroup  $U$  of  $L$  such that the centralizer of  $U$  in  $L$ , viz.  $\{x \in L \mid xu = ux \text{ for all } u \in U\}$ , does not contain any compact subgroup of positive dimension.

**Theorem 1.2.** *Let  $G$  be an exponential Lie group and  $R$  be the radical of  $G$ . Let  $S = G/R$  and suppose that  $S$  satisfies condition  $\mathfrak{U}$ . Then  $R$  is an exponential Lie group.*

In the context of the theorem it would be of interest to know when a semisimple Lie group satisfies condition  $\mathfrak{U}$ . Let  $S$  be a semisimple Lie group and let  $S = KAN$  be an Iwasawa decomposition of  $S$ , namely  $K, A$  and  $N$  are closed connected subgroups of  $S$ , and if  $S^*$  is the adjoint group of  $S$  and  $\text{Ad} : S \rightarrow S^*$  is the adjoint representatiton, then  $\text{Ad}(K)$  is a maximal compact subgroup of  $S^*$ ,  $\text{Ad}(A)$  is a maximal connected subgroup whose action on the Lie algebra is diagonalisable over  $\mathbb{R}$ , and  $N$  is a simply connected nilpotent Lie subgroup of  $S$  normalised by  $A$  (see for example [7]); we note that  $K$  contains the center of  $S$  and  $K$  itself is not compact when the center is infinite, as in the case of some nonlinear semisimple Lie

groups. Let  $M$  be the centraliser of  $A$  in  $K$ . Then  $M$  is a closed (not necessarily connected) subgroup of  $S$ . Also  $M$  normalises  $N$  and  $P = MAN$  is a minimal parabolic subgroup of  $S$ .

With regard to condition  $\mathfrak{U}$  we prove the following.

**Theorem 1.3.** *Let the notation be as above and suppose that  $M$  is abelian. Then  $S$  satisfies condition  $\mathfrak{U}$ . In particular if  $S$  is a  $\mathbb{R}$ -split semisimple Lie group or a complex semisimple Lie group then it satisfies condition  $\mathfrak{U}$ .*

**Remark 1.4.** It has also been possible to prove (without reference to  $M$  being abelian) that if  $N$  contains an  $\mathbb{R}$ -regular unipotent element of  $S$ , in the sense of [1], which is not centralised by any compact subgroup of  $P$  of positive dimension then  $S$  satisfies condition  $\mathfrak{U}$ . The condition holds when  $M$  is abelian. It has however not been possible to ascertain whether there are semisimple Lie groups  $S$  with  $M$  nonabelian for which this holds. We shall therefore not go into the technical details of this possible generalisation, at this stage.

When  $S$  is the group of  $\mathbb{R}$ -elements of a semisimple algebraic group  $\underline{S}$  defined over  $\mathbb{R}$  (or the connected component of the identity in such a group) the condition of  $M$  being abelian is equivalent to  $\underline{S}$  being a quasi-split group, namely one admitting a Borel subgroup defined over  $\mathbb{R}$  (see [8] and [2]). Thus Theorem 1.3 applies in this case; we note however that  $S$  as in Theorem 1.3 need not be linear.

Theorems 1.2 and 1.3 together imply the following, which yields in particular the statement from [9] recalled above.

**Corollary 1.5.** *Let  $G$  be an exponential Lie group,  $R$  be the radical of  $G$  and  $S = G/R$ . Suppose  $S$  satisfies the condition as in Theorem 1.3. Then  $R$  is an exponential Lie group. In particular the center of  $R$  is connected.*

We note that for condition  $\mathfrak{U}$  to hold for a semisimple Lie group it has to be noncompact. Also, not all noncompact semisimple Lie groups satisfy condition  $\mathfrak{U}$ ; this can be readily verified for the group  $SU(n, 1)$  for instance, which is in fact one of the exponential Lie groups (see [6],[11]). The question whether a Lie group  $G$  being exponential implies that its radical is exponential remains open in the cases when  $G/R$  is one of these semisimple Lie groups.

Towards proving Theorem 1.2 we also prove a result on exponential elements in solvable Lie groups, which may be of independent interest. The question of exponentiality of solvable Lie groups was considered earlier in [10] (and earlier papers cited there), but the focus there has been on criteria for the group to be exponential, namely *all* elements being exponential. On the other hand our result below, Theorem 2.2, is about exponentiality of elements from certain subsets. Moreover, the argument in [10] involves the theory of Cartan subgroups, which is technically more intricate; our argument is based on more elementary considerations.

The paper is organised as follows. The result on exponential elements in solvable Lie groups, Theorem 2.2, will be taken up in §2, and in §3 the results are applied to prove Theorem 1.2 and some related results. In §4 we discuss when condition  $\mathfrak{U}$  is satisfied, and prove Theorem 1.3.

## 2 Exponential elements in solvable Lie groups

Let  $H$  be a connected solvable Lie group and  $N$  be a nilpotent simply connected closed normal Lie subgroup of  $H$  such that  $H/N$  is abelian. We denote by  $\mathfrak{N}(H)$  the class of closed connected normal subgroups of  $H$  contained in  $N$ . We shall be considering pairs of the form  $(M, M')$ , with  $M, M' \in \mathfrak{N}(H)$ , and  $M' \subset M$ , and for brevity we shall refer to such a pair as a *subquotient* (of  $N$ , with respect to the action of  $H$ ). A subquotient  $(M, M')$  is said to be abelian if  $M/M'$  is abelian; as  $N$  is simply connected, in this case  $M/M'$  is a vector space over  $\mathbb{R}$ . The conjugation action of  $H$  on itself induces an action on each  $M \in \mathfrak{N}(H)$ , by restriction, and hence on all subquotients of  $H$ . An abelian subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , will be called an *irreducible subquotient* if the  $H$ -action on  $M/M'$  is irreducible. We note that for any irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , the action of  $N$  on  $M/M'$  is trivial, and hence the action factors to an irreducible linear action of  $A$  on  $M/M'$ .

**Definition 2.1.** Let the notation be as above. Let  $a \in A = H/N$  and  $B$  be a one-parameter subgroup of  $A$  containing  $a$ . We say that the pair  $(a, B)$  is of *type  $\mathcal{E}$*  if the following holds: if for an irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , the action of  $a$  on  $M/M'$  is trivial, then the action of  $B$  on  $M/M'$  is trivial; the element  $a$  is said to be *type  $\mathcal{E}$*  if there exists a one-parameter subgroup  $B$  of  $A$  such that the pair  $(a, B)$  is of type  $\mathcal{E}$ .

We prove the following.

**Theorem 2.2.** *Let  $H$  be a connected solvable Lie group and  $N$  be a nilpotent simply connected closed normal Lie subgroup of  $H$  such that  $H/N$  is abelian. Let  $A = H/N$ . Let  $x \in H$  and  $a = xN \in A$ . Then the following conditions are equivalent:*

- i)  $a$  is of type  $\mathcal{E}$ .*
- ii) for every  $n \in N$ , the element  $xn$  is exponential in  $H$ .*

We first consider the following special case, in which we prove some more precise results. By a *vector subgroup* we mean a subgroup which is topologically isomorphic to  $\mathbb{R}^d$  for some  $d \geq 1$  (with respect to the induced topology). A vector subgroup will be considered equipped with its canonical structure as a vector space over  $\mathbb{R}$ .

**Proposition 2.3.** *Let  $L$  be a connected Lie group. Let  $V$  be a vector subgroup of  $L$  and  $P$  be a one-parameter subgroup normalizing  $V$ , and such that the conjugation action of  $P$  on  $V$  is irreducible and nontrivial. Let  $H = PV$ . Let  $p \in P$  be nontrivial and let  $\sigma(p) : V \rightarrow V$  denote the conjugation action of  $p$  on  $V$ . Then the following holds:*

- i) Every one-parameter subgroup of  $L$  contained in  $H$  is either contained in  $V$  or is of the form  $wPw^{-1}$  for some  $w \in V$ .*
- ii) If  $\sigma(p)$  is nontrivial then for any  $v \in V$ ,  $pv$  is contained in a unique one-parameter subgroup contained in  $H$ .*
- iii) If  $\sigma(p)$  is trivial, then for  $v \in V$ ,  $v \neq 0$  (the zero element in  $V$ ),  $pv$  is not contained in any one-parameter subgroup of  $H$ .*

*Proof.* At the outset we note that the subgroup  $H$  as in the hypothesis need be a closed subgroup of  $L$ . It is however a Lie subgroup whose Lie algebra is the sum of the Lie subalgebras of  $P$  (which is one-dimensional) and  $V$ . Let  $\mathfrak{H}$  be the Lie algebra of  $H$ ; we shall realize it as  $\mathbb{R}\xi \oplus V$ , where  $\xi$  is a generator of the Lie algebra of  $P$  and the vector space  $V$  is identified with its Lie algebra. We now prove the statements as in the Proposition.

i) For  $w \in V$  the Lie subalgebra of  $\mathfrak{H}$  corresponding to the one-parameter subgroup  $wPw^{-1}$  is spanned by  $\xi + (\theta(w) - w)$ , where  $\theta : V \rightarrow V$  is given by  $\theta(u) = \frac{dp_{-t}(u)}{dt}$  for all  $u \in V$ . We see that as the  $P$ -action on  $V$  is nontrivial and irreducible the map  $w \mapsto \theta(w) - w$  is surjective, and hence all elements of the form  $\xi + v$ ,  $v \in V$ , are among the tangents to  $wPw^{-1}$ ,  $w \in V$ . Since scalar multiples of these cover all elements that are not contained in  $V$ , it follows that every one-parameter subgroup which is not contained in  $V$  is of the form  $wPw^{-1}$  (upto scaling of the parameter). This proves (i).

ii) Let  $p \in P$  be such that  $\sigma(p)$  is nontrivial. As the  $P$ -action on  $V$  is irreducible and nontrivial this implies in particular that 1 is not an eigenvalue of  $\sigma(p)$ . Hence  $\sigma(p)^{-1} - I$ , where  $I$  is the identity transformation, is invertible. Now let  $v \in V$  be given. Then there exists  $w \in V$  such that  $v = \sigma(p)^{-1}(w) - w$ . Hence, in  $H$ ,  $v = (p^{-1}wp)w^{-1}$ . Thus  $pv = wpw^{-1}$ , and it is contained in the one-parameter subgroup  $wPw^{-1}$ .

We note that if  $pv$  is contained in  $wPw^{-1}$  for some  $w \in V$  then in fact  $pv = wpw^{-1}$ ; if  $q \in P$  is such that  $pv = qwq^{-1} = q(q^{-1}wqw^{-1}) \in qV$ , then  $p^{-1}q \in V$  and hence  $pv = qwq^{-1} = (wpw^{-1})(wp^{-1}qw^{-1}) = wpw^{-1}$ . Now if  $pv$  is contained in  $w_1Pw_1^{-1}$  and  $w_2Pw_2^{-1}$ ,  $w_1, w_2 \in V$ , we have  $w_1pw_1^{-1} = w_2pw_2^{-1}$ , which means that  $w_2^{-1}w_1$  (or  $w_1 - w_2$  is additive notation) is fixed by  $\sigma(p)$ . Since  $\sigma(p)$  is nontrivial and the  $P$ -action is irreducible this implies that  $w_1 = w_2$ . Thus one-parameter subgroup containing  $pv$  is unique. This proves (ii).

iii) Now let  $p \in P$  be such that  $\sigma(p)$  is trivial. Let  $v \in V$ ,  $v \neq 0$  and suppose that  $pv$  is contained in  $wPw^{-1}$  for some  $w \in V$ . Then, as before, we

have  $pv = wpw^{-1}$ . Hence  $v = p^{-1}wpw^{-1} = \sigma(p)^{-1}(w) - w = 0$ , contradicting the assumption that  $v \neq 0$ . Hence  $pv$  is not contained in any one-parameter subgroup of  $H$ . This proves (iii).  $\square$

In the sequel it will be convenient to use the following terminology.

**Definition 2.4.** Given a Lie group  $L$ , a closed normal subgroup  $M$  of  $L$  with  $\eta : L \rightarrow L/M$  the canonical quotient map, and a one-parameter subgroups  $\Psi$  of  $L/M$ , by a *lift* of  $\Psi$  in  $L$  we mean a one-parameter subgroup  $\Phi$  of  $L$  such that  $\eta(\Phi) = \Psi$ .

**Proposition 2.5.** *Let the notations  $H, N, A, x$  and  $a$  be as in Theorem 2.2. Let  $B$  be a one-parameter subgroup of  $A$  containing  $a$ , such that  $(a, B)$  is of type  $\mathcal{E}$ . Then for any  $n \in N$  there exists a lift of  $B$  containing  $xn$ . Moreover, the collection of such lifts is countable.*

*Proof.* We shall show that if  $(M, M')$ , where  $M, M' \in \mathfrak{N}(H)$ , is an irreducible subquotient then the following holds: given a one-parameter subgroup  $\Psi$  of  $H/M$  which is a lift of  $B$  and  $y \in H$  such that  $yN = a$ , there exist at least one and at most countably many lifts  $\Phi$  of  $\Psi$  in  $H/M'$  containing  $yM'$ . Applying this to successive pairs from a sequence  $N_0, N_1, \dots, N_k$  in  $\mathfrak{N}(H)$  such that  $N_0 = N$ ,  $N_k$  is trivial, and for all  $j = 0, \dots, k-1$ ,  $N_{j+1} \subset N_j$  and  $(N_j, N_{j+1})$  is an irreducible subquotient, yields the statement as in the proposition.

Now let  $(M, M')$  as above, a one-parameter subgroup  $\Psi$  of  $H/M$  which is a lift of  $B$ , and  $y \in H$  such that  $yM = a$  be given. Let  $P$  be a lift of  $\Psi$  in  $H/M'$ . We now apply Proposition 2.3 to  $P(M/M')$  (the product of  $P$  and  $M/M'$  in  $H/M'$ ), with  $v = yM'$ . We note that any one-parameter subgroup of  $H/M'$  which is a lift of  $\Psi$  is contained in  $P(M/M')$ , thus it suffices to show that there exists a lift  $\Phi$  of  $\Psi$  in  $P(M/M')$  containing  $yM'$ . If the action of  $a$  on  $M/M'$  is nontrivial then this is assured by Proposition 2.3(ii). On the other hand if the action of  $a$  on  $M/M'$  is trivial then by hypothesis the action of  $B$  is also trivial, and since  $P$  is a lift of  $B$  this implies that  $P(M/M')$  is an abelian Lie group, in this case the assertion is obvious. This proves the Proposition.  $\square$

**Proposition 2.6.** *Let  $H, N, A, x$  and  $a = xN$  be as in Theorem 2.2 and let  $B$  be a one-parameter subgroup of  $A$  containing  $a$ . Let  $M \in \mathfrak{N}(H)$  be an abelian subgroup such that the  $A$ -action on  $M$  is irreducible and the following holds:*

*i) for the group  $H/M$ , with  $A$  being viewed canonically as  $(H/M)/(N/M)$ , the pair  $(a, B)$  is of type  $\mathcal{E}$ .*

*ii) on  $M$  the action of  $a$  is trivial but the action of  $B$  is not trivial.*

*Let  $Q$  be a lift of  $B$  in  $H/M$  and  $y \in H$  be such that  $yN = a$ . Then there exist a unique  $m \in M$  such that  $ym$  is contained in a lift of  $Q$ .*

*Proof.* Let  $P$  be a lift of  $Q$  in  $H$ . Let  $p \in P$  be such that  $pM = yM$ , and  $m \in M$  such that  $p = ym$ . Then  $pN = yN = a$ . By condition (ii) in the hypothesis and Proposition 2.3(iii), applied to the subgroup  $PM$ , we get that  $pm'$  is not contained in a lift of  $Q$  for any nontrivial element  $m'$  of  $M$ . Thus  $m$  is the only element of  $M$  such that  $ym$  is contained in a lift of  $Q$ .  $\square$

**Proposition 2.7.** *Let  $H, N, A, x$  and  $a = xN$  be as in Theorem 2.2 and let  $B$  be a one-parameter subgroup of  $A$  containing  $a$ . Suppose that  $(a, B)$  is not of type  $\mathcal{E}$ . Let  $E = \{n \in N \mid xn \text{ is contained in a lift of } B \text{ in } H\}$ . Then  $E$  has 0 Haar measure in  $N$ .*

*Proof.* At the outset we note that the set of exponential elements in a Lie group is a Borel subset, and hence it follows that  $E$  as in the hypothesis is a Borel subset of  $N$ . As  $(a, B)$  is not of type  $\mathcal{E}$  there exists an irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , such that the  $a$ -action on  $M/M'$  is trivial and the  $B$  action is nontrivial, and we may without loss of generality assume  $M$  to be of maximal possible dimension among such pairs. To prove the proposition it suffices to show that  $EM'/M'$  has zero Haar measure in  $N/M'$ , and hence passing to  $H/M'$ , we may without loss of generality assume  $M'$  to be trivial. Thus  $M$  is a normal vector subgroup of  $H$ . By the maximality condition on  $M$ ,  $(a, B)$  is of type  $\mathcal{E}$  for  $H/M$ . Hence by Proposition 2.5 for any  $n \in N$  there exist only countably many lifts of  $B$  in  $H/M$  containing  $xnM$ . Let  $n \in N$  and  $Q$  be any lift of  $B$  in  $H/M$  containing  $xnM$ . Let  $P$  be a lift of  $Q$  in  $H$  and  $m \in M$  be such that  $xnm \in P$ . We now apply Proposition 2.3(iii), with  $M$  in place of  $V$  there. The choice of  $M$  as above shows that the conjugation action of  $xnm$  on  $M$  is trivial but the action of  $P$  is not trivial. By Proposition 2.3(iii) therefore  $xnm$  is the only element in  $xnM$  which is contained in a lift of  $Q$ . Since there are only countably many lifts  $Q$  of  $B$  containing  $xnM$ , this shows that there are only countably many  $m$  in  $M$  such that  $xnm$  is contained in a lift of  $B$  in  $H$ . In other words,  $E$  as in the hypothesis intersects each coset  $xnM$ , where  $n \in N$ , in only countably many points. It follows that its Haar measure of  $E$  must be 0.

*Proof of Theorem 2.2:* Statement (i)  $\implies$  (ii) follows immediately from (the existence statement in) Proposition 2.5. We now prove the converse. We suppose that condition (ii) holds, but  $a$  is not of type  $\mathcal{E}$ , and arrive at a contradiction.

Let  $\mathcal{C}$  be the class of  $L \in \mathfrak{N}(H)$ , such that  $xN/L$  is not of type  $\mathcal{E}$  for the group  $H/L$ . Then  $\mathcal{C}$  is nonempty since by assumption  $a$  is not of type  $\mathcal{E}$  for  $H$ , so  $\mathcal{C}$  contains the trivial subgroup. Let  $L$  be an element of  $\mathcal{C}$  with maximum possible dimension; we note that  $L$  is a proper subgroup of  $N$ , since the action of  $A$  on  $N/L$  has to be nontrivial. For notational convenience we shall consider  $H/L$  as  $H$ , and respectively  $N/L$  as  $N$ , and  $xL$  as  $x$ . Then in the modified notation we have that  $xn$  is exponential in  $H$  for all  $n \in N$ ,  $xN$  is not of type  $\mathcal{E}$ , but  $xN/M$  is of type  $\mathcal{E}$  for every  $M \in \mathfrak{N}(H)$  of positive dimension. We fix a  $M \in \mathfrak{N}(H)$  of

positive dimension such that the action of  $A = H/N$  on  $M$  is irreducible. Let  $a = xN$ . Then the action of  $a$  on  $M$  is trivial since otherwise  $a$  would be of type  $\mathcal{E}$  under the conditions as above.

By hypothesis, for each  $n \in N$  there exists a one-parameter subgroup containing  $xn$ . We note that the set of one-parameter subgroups of  $A$  containing  $a = xN$  is countable, say  $\{B_j\}$  with  $j$  running over a countable set; it can be a singleton set as would be the case when  $A$  is simply connected.

By Proposition 2.7, applied to  $H/M$  there exists  $n \in N$  such that  $xnM$  is not contained in a lift of  $B_j$  for any  $j$  such that  $(a, B_j)$  is not of type  $\mathcal{E}$  for  $H/M$ ; indeed the set of such  $n$  is a set of full Haar measure. We fix such  $n$  and consider the elements  $xnm$ ,  $m \in M$ . By hypothesis each of them is exponential and hence is contained in a lift of  $B_j$  for some  $j$ , and by the choice of  $n$ , the  $j$  must be such that  $(a, B_j)$  is of type  $\mathcal{E}$  for  $N/M$ . Hence to prove the theorem it suffices to show that the action of one of these on  $M$  is trivial. Consider any  $B_j$  such that  $(a, B_j)$  is of type  $\mathcal{E}$  for  $H/M$ . Then by Proposition 2.5 for any  $j$  there exist only countably many lifts of  $B_j$  to  $H/M$  containing  $xnM$ . Let  $Q$  be any lift of  $B_j$  containing  $xnM$ . If the action of  $B_j$  on  $M$  is nontrivial then by Proposition 2.6 there exists a unique  $m \in M$  such that  $xnm$  is contained in a lift of  $Q$  in  $H$ . It follows therefore that if the action of  $B_j$  on  $M$  is nontrivial then the set of  $m \in M$  for which  $xnm$  is contained in a lift of  $B_j$  is countable. But this is a contradiction since in fact since every  $xnm$ ,  $m \in M$ , is contained in a one-parameter subgroup of  $H$ , which necessarily has to be a lift of some  $B_j$  such that  $(a, B_j)$  is of type  $\mathcal{E}$  on  $H/M$ . The contradiction shows that  $a$  must indeed be of type  $\mathcal{E}$ , which proves the theorem.  $\square$

**Remark 2.8.** The criterion in Theorem 2.2 involves that for  $a \in A$  there exists a *common* one-parameter subgroup  $B$  such that for every irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , such that if the action of  $a$  on  $M/M'$  is trivial the action of  $B$  is also trivial. It does not suffice to have such a one-parameter for each  $(M, M')$  individually, and dependent on it. This is illustrated by the following example: Let  $H$  be the semi-direct product of  $\mathbb{T}^2 = \{(\rho_1, \rho_2) \mid \rho_1, \rho_2 \in \mathbb{C}, |\rho_1| = |\rho_2| = 1\}$  and  $\mathbb{C}^2 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}\}$ , with the conjugation action of  $(\rho_1, \rho_2)$  given by  $(z_1, z_2) \mapsto (\rho_1 \rho_2 z_1, \rho_1 \rho_2^{-1} z_2)$ . Then  $N$  and  $A$  as in the earlier notation may be seen to correspond to  $\mathbb{C}^2$  and  $\mathbb{T}^2$  respectively, (together with the actions involved). The irreducible subquotient actions of  $A$  on  $N$  are seen to correspond to actions on the two copies of  $\mathbb{C}$ , corresponding to the two coordinates. Let  $a = (-1, -1) \in \mathbb{T}^2$ . Then the action of  $a$  on  $\mathbb{C}^2$  is trivial. We see that there are one-parameter subgroups  $B_1$  and  $B_2$  of  $\mathbb{T}^2$  containing  $a$  acting trivially on the subquotients corresponding to the first and second coordinates respectively, but there is no one-parameter subgroup containing  $a$  and acting trivially on both coordinates. Thus the condition of Theorem 2.2 is not satisfied. Hence by the Theorem the group is not exponential, as may also be checked directly.



Theorem 2.2 can be used to deduce the following property, known earlier ([10], Corollary 3.18).

**Corollary 2.9.** *Let  $H$  be an exponential solvable Lie group. Then the center of  $H$  is connected.*

*Proof.* Let  $N$  be the nilradical of  $H$  and  $A = H/N$ . Let  $z$  be an element contained in the center of  $H$  and  $a = zN \in A$ . Since  $H$  is exponential  $xn$  is exponential for all  $n \in N$ . Also,  $z$  being contained in the center, the action of  $a$  on any irreducible subquotient is trivial. Hence by Theorem 2.2 there exists a one-parameter subgroup  $B$  of  $A$  containing  $a$  such that the action of  $B$  on any irreducible subquotient is trivial. Let  $L$  be the subgroup of  $H$  containing  $N$  and such that  $B = L/N$ . Then the preceding observation implies that  $L$  is a connected nilpotent Lie group. Since  $N$  is the nilradical we get that  $L = N$ . Therefore  $z$  is contained in the center of  $N$ , say  $Z$ . Now  $Z$  is a connected abelian Lie group, and the center of  $H$  is precisely the set of fixed points of the action of  $A$  on  $Z$ , induced by the  $A$ -action on  $N$ . It can be seen that as  $A$  is connected, the set of fixed points is a connected subgroup of  $Z$ . Hence the center of  $H$  is connected.  $\square$

We also deduce the following characterization of exponential Lie group, which is in a sense the main nontechnical part in the characterization of exponentiality of solvable Lie groups in Theorem 3.17 of [10]. We follow the terminology as in [10]; however the symbols chosen are different, to avoid clash with the notation in the rest of this paper.

**Corollary 2.10.** *Let  $H$  be a connected solvable Lie group,  $\mathfrak{H}$  be the Lie algebra of  $H$ , and  $C$  be a Cartan subgroup of  $H$ . Then  $H$  is an exponential Lie group if and only if for all nilpotent elements  $\nu \in \mathfrak{H}$  the centralizer of  $\nu$  in  $C$ , namely  $Z_C(\nu) := \{y \in C \mid \text{Ad}(y)(\nu) = \nu\}$ , is connected.*

*Proof.* Let  $N$  be the nilradical of  $H$  and  $\mathfrak{N}$  be the Lie subalgebra corresponding to  $N$ . Then we have  $H = CN$ . As  $\mathfrak{H}$  is solvable, all nilpotent elements in  $\mathfrak{H}$  are contained in  $\mathfrak{N}$ . Also,  $\mathfrak{N}$  has a decomposition as  $\bigoplus_{s \in S} \mathfrak{N}_s$ , with  $S$  an indexing set, such that the following holds: for each  $s \in S$ ,  $\mathfrak{N}_s$  is  $\text{Ad } C$ -invariant, for each  $y \in C$  the eigenvalues of the restriction of  $\text{Ad } y$  to  $\mathfrak{N}_s$  consist of a complex conjugate pair, say  $\lambda(y, s)$  and  $\overline{\lambda(y, s)}$  (only one when real), and there exists an order on  $S$ , denoted by  $\geq$ , such that for any  $t \in S$ ,  $\sum_{s \geq t} \mathfrak{N}_s$  is a Lie ideal in  $\mathfrak{H}$ ; more precise relations can be written down for commutators of the  $\mathfrak{N}_s$ 's but we do not need that here. We note also that  $C$  is a connected nilpotent Lie group and hence given  $y \in C$  and a one-parameter subgroup  $B'$  of  $CN/N$  containing  $yN$  there exists a one-parameter  $B$  of  $C$  containing  $y$  such that  $B' = BN/N$ .

Now suppose that  $H$  is exponential and let a nilpotent  $\nu$  element be given. Then we have  $\nu = \sum_{s \in S} \nu_s$ , with  $\nu_s \in \mathfrak{N}_s$  for all  $s \in S$ . Let  $y \in C$  and  $a = yN/N$ .

By Theorem 2.2, together with one of the observations above, there exists a one-parameter subgroup  $B$  of  $C$  containing  $y$  such that for any irreducible subquotient  $(M, M')$  such that the action of  $y$  on  $M/M'$  is trivial, the action of  $B$  on  $M/M'$  is also trivial. Now,  $\text{Ad } y(\nu) = \nu$  if and only if  $\lambda(y, s) = 1$  for all  $s$  such that  $\nu_s \neq 0$ . Consider any  $t \in S$  such that  $\lambda(y, t) = 1$ . Let  $M$  be the closed subgroup with Lie algebra  $\sum_{s \geq t} \mathfrak{N}_s$  and  $M'$  be a closed connected normal subgroup of  $H$ , properly contained in  $M$ , such that the Lie algebra of  $M'$  contains  $\mathfrak{N}_s$ , for all  $s > t$  (that is  $s \geq t$  and  $s \neq t$ ), and  $(M, M')$  is an irreducible quotient. As  $\lambda(y, t) = 1$  it follows that the action of  $y$  on  $M/M'$  is trivial. Hence the action of  $B$  on  $M/M'$  is also trivial and in turn  $\lambda(b, t) = 1$  for all  $b \in B$ . Thus we get that  $\text{Ad } b(\nu) = \nu$  for all  $b \in B$ . This shows that the centralizer of  $\nu$  in  $C$  is connected.

Now suppose that  $Z_C(\nu)$  is connected for all  $\nu \in \mathfrak{N}$ . Let  $x \in H$  be given and let  $a = xN \in H/N$ . Since  $H = CN$  there exists  $y \in C$  such that  $a = yN/N$ . Let  $S' = \{s \in S \mid \lambda(y, s) = 1\}$ . There exists an element  $\nu = \sum_{s \in S'} \nu_s$ , such that each  $\nu_s$ ,  $s \in S'$ , is a nonzero element of  $\mathfrak{N}_s$  such that  $\text{Ad } y(\nu_s) = \nu_s$ . In particular  $\nu$  is fixed by  $\text{Ad } y$ , and since  $Z_C(\nu)$  is connected and nilpotent we get that there exists a one-parameter subgroup  $B$  of  $C$  containing  $y$  such that  $\text{Ad } b(\nu) = \nu$  for all  $b \in B$ . Hence  $\text{Ad } b(\nu_s) = \nu_s$ , and in turn  $\lambda(b, s) = 1$ , for all  $s \in S'$  and  $b \in B$ . Now let  $B' = BN/N$ . Then  $B'$  is a one-parameter subgroup of  $H/N$  containing  $a$ ; we shall show that  $(a, B')$  is of type  $\mathcal{E}$ .

Let  $(M, M')$  be an irreducible subquotient such that the action of  $a$  on  $M/M'$  is trivial. Then there exists  $t \in S'$  such that  $M$  is contained in  $\sum_{s \geq t} \mathfrak{N}_s$  and  $M'$  contains  $\sum_{s > t} \mathfrak{N}_s$ . Since  $\lambda(b, s) = 1$  for all  $s \in S'$  and  $b \in B$ , it follows that the action of  $B' = BN/N$  on  $M/M'$  is trivial. Thus  $(a, B')$  is of type  $\mathcal{E}$  and hence by Theorem 2.2  $x$  is exponential in  $H$ . Therefore  $H$  is an exponential Lie group.  $\square$

### 3 Proof of Theorem 1.2

We shall now deduce Theorem 1.2; we follow the notations  $G$ ,  $R$ , and  $S$  as in the statement of the theorem in §1. Since  $S$  satisfies condition  $\mathfrak{U}$  there exists a unipotent one-parameter subgroup  $U$  of  $S$  such that the centralizer of  $U$  in  $S$  contains no compact subgroup of positive dimension. We note the following.

**Lemma 3.1.** *Let  $S$  and  $U$  be as above. Let  $u$  be a nontrivial element of  $U$ . Then  $U$  is the only one-parameter subgroup of  $S$  containing  $u$ .*

*Proof.* We may assume  $U = \{u_t\}$  and  $u = u_1$ . It suffices to show that if  $\{x_t\}$  is any one-parameter subgroup of  $S$  such that  $x_1 = u$  then  $x_t = u_t$  for all  $t$ . Let  $s \in \mathbb{R}$  be arbitrary. Since  $x_s$  commutes with  $x_1 = u$  and  $\{u_t\}$  is a unipotent one-parameter subgroup it follows that  $x_s$  commutes with  $u_t$  for all  $t \in \mathbb{R}$ . Since this holds for all  $s \in \mathbb{R}$  it follows that  $\{x_{-t}u_t\}$  is a one-parameter subgroup. Moreover, since

$x_{-1}u_1 = u^{-1}u = e$ , the identity element,  $\{x_{-t}u_t\}$  is a compact subgroup. Since it is contained in the centraliser of  $U$ , the condition on  $U$  implies that the subgroup is trivial, namely  $x_t = u_t$  for all  $t$ . This proves the Lemma.  $\square$

Now let  $\eta : G \rightarrow S$  be the canonical quotient map of  $G$  onto  $S$ , and let  $H = \eta^{-1}(U)$ . We note that  $U$  is a closed connected subgroup of  $S$  and hence  $H$  is a connected Lie group. Since  $N$  is the nilradical of  $G$ ,  $G/N$  is reductive, and in particular it follows that  $H/N$  is abelian; in particular  $H$  is a solvable Lie group.

We now first prove the following.

**Proposition 3.2.** *Any  $h \in H$  which is not contained in  $R$  is exponential in  $H$ .*

*Proof.* Let  $h \in H \setminus R$  be given. Since  $G$  is exponential there exists a one-parameter subgroup  $P$  of  $G$  containing  $h$ . Consider the one-parameter subgroup  $\eta(P)$ . We note that  $\eta(h)$  is a nontrivial element of  $U$ . Hence by Lemma 3.1 we have  $\eta(P) = U$ . Thus  $P$  is contained in  $\eta^{-1}(U) = H$ . Therefore  $h$  is exponential in  $H$ .  $\square$

*Proof of Theorem 1.2:* We first note that in proving the theorem  $N$  may be assumed to be simply connected: Let  $C$  be maximal compact subgroup of  $N$ . Then  $C$  is a connected subgroup contained in the centre of  $G$ . We note that the hypothesis of the theorem holds for  $G/C$ , and upholding that the desired conclusion for  $G/C$  (namely showing that  $R/C$  is exponential) implies that  $R$  is exponential, as desired. We may therefore assume without loss of generality that  $C$  is trivial. Equivalently this means that  $N$  is simply connected.

Now let  $x \in R$  be given. Let  $H$  be the solvable group introduced above and  $y = ux \in H$ , where  $u$  is a nontrivial element of  $U$ . We note that since  $H/R$  is topologically isomorphic to  $\mathbb{R}$ , to prove that  $x$  is exponential in  $R$  it suffices to prove that it is exponential in  $H$ . Now let  $A = H/N$  and  $a, a'$  be the elements  $a = xN$  and  $a' = yN$ . By Proposition 3.2  $yn$  is exponential in  $H$  for all  $n \in N$ . Hence by Theorem 2.2, there exists a one-parameter subgroup  $B'$  of  $A$  containing  $a'$  such that for any irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , for which the action of  $a'$  on  $M/M'$  is trivial, the action of  $B'$  on  $M/M'$  is also trivial. Let  $B' = \{b'_t\}$  and  $U = \{u_t\}$ , with  $b'_1 = a'$  and  $u_1 = u$ . Let  $B = \{b_t\}$  be the one-parameter subgroup of  $A$  defined by  $b_t = (u_{-t}N)b'_t$  for all  $t \in \mathbb{R}$ ; in particular  $b_1 = u^{-1}a' = a$ . We note that as  $U$  is a unipotent one-parameter subgroup of  $S$ , the action of  $U$  on any irreducible subquotient  $M/M'$  as above is trivial. Hence on any irreducible subquotient the actions of  $b_t$  and  $b'_t$  coincide for each  $t \in \mathbb{R}$ . Therefore we get that for any irreducible subquotient  $(M, M')$ ,  $M, M' \in \mathfrak{N}(H)$ , for which the action of  $a$  on  $M/M'$  is trivial, the action of  $B$  on  $M/M'$  is trivial. Thus condition (i) of Theorem 2.2 holds for  $x$ , and the theorem implies that  $xn$  is exponential for all  $n \in N$ ; in particular  $x$  is exponential in  $H$ , and hence, as seen above, also in  $R$ . This proves the theorem.  $\square$

**Corollary 3.3.** *Let  $G$  be a Lie group as in Theorem 1.2 and  $R$  be its radical. Let*

$Z(G)$  and  $Z(R)$  be the centers of  $G$  and  $R$  respectively. Then  $Z(R)$  and  $Z(G) \cap R$  are connected.

*Proof.* By Corollary 2.9  $Z(R)$  is connected. Clearly  $Z(G) \cap R$  is the set of fixed points of the canonical action of  $G/R$  on  $Z(R)$ , induced by the conjugation action of  $G$ . As  $Z(R)$  is abelian and  $G/R$  is connected it follows that the set of fixed points is a connected subgroup.  $\square$

## 4 Groups satisfying condition $\mathfrak{U}$

In this section we consider semisimple Lie groups  $S$  satisfying condition  $\mathfrak{U}$  and prove Theorem 1.3. We shall follow the notation as in §1; we recall in particular that  $M$  denotes the subgroup consisting of all elements in a maximal compact subgroup  $K$  of  $S$ , commuting with all elements in a maximal subgroup  $\text{Ad}$ -diagonalisable over  $\mathbb{R}$ .

**Remark 4.1.** We note that when  $M$  is abelian there exists a unipotent one-parameter subgroup whose centraliser is contained in the center of  $S$ . It suffices to see this in the case when  $S$  has trivial center, so  $S$  may be taken to be the group of  $\mathbb{R}$ -elements of a semisimple algebraic group defined over  $\mathbb{R}$ . In this case  $M$  being abelian is equivalent to the group being quasi-split, namely that there exists a Borel subgroup defined over  $\mathbb{R}$ . Under that condition it follows from [8], Proposition 5.1 (see also [2] for a generalisation of the result to non-semisimple algebraic groups) and [1] and Corollary 5 (page 114) that there exists  $u \in N$  such that the centraliser of  $u$  in  $S$  is contained in  $P$ . On the other the conjugation action of  $M$  action on  $N$  has no fixed points, so the centraliser of  $u$ , and hence of any one-parameter subgroup  $U$  containing  $u$ , has no compact subgroup of positive dimension. Hence  $S$  satisfies condition  $\mathfrak{U}$ .

*Proof of Theorem 1.3:* Since  $M$  is abelian by Remark 4.1 there exists a unipotent one-parameter subgroup, say  $U$ , in  $S$  whose centraliser is contained in the (discrete) center of  $S$ ; in particular the centraliser of  $U$  in  $S$  contains no subgroup of positive dimension. Hence condition  $\mathfrak{U}$  is satisfied for  $S$ .

For split semisimple Lie groups the subgroup  $M$  as above is trivial, and for complex Lie groups it is abelian. It therefore follows from the above results the condition  $\mathfrak{U}$  holds in these cases.  $\square$

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